

# Spinfoam cosmology with the proper vertex amplitude

Ilya Vilensky\*

*Department of Physics, Florida Atlantic University, Boca Raton, Florida, USA*

## Abstract

The proper vertex amplitude is derived from the EPRL vertex by restricting to a single gravitational sector in order to achieve the correct semi-classical behaviour. We apply the proper vertex to calculate a cosmological transition amplitude that can be viewed as the Hartle-Hawking wavefunction. To perform this calculation we deduce the integral form of the proper vertex and use extended stationary phase methods to estimate the large-volume limit. We show that the resulting amplitude satisfies an operator constraint whose classical analogue is the Hamiltonian constraint of the Friedmann-Robertson-Walker cosmology. We find that the constraint dynamically selects the relevant family of coherent states and demonstrate a similar dynamic selection in standard quantum mechanics.

## I. INTRODUCTION

Spinfoam models provide a path integral description of the dynamics of loop quantum gravity (LQG), a proposed theory of quantum gravity. The most widely studied model is the Engle-Pereira-Rovelli-Livine (EPRL) vertex amplitude [1–3]. However, it has been pointed out that this model fails to select a single gravitational sector [4] which may lead to unphysical contributions in the semi-classical limit from configuration histories that do not satisfy the classical equations of motion. A proposed modification of the vertex amplitude that resolves this issue by introducing a quantum mechanical restriction to a single gravitational sector has been developed under the name of the ‘proper’ vertex amplitude [4–7].

One of the most important tasks before any theory of quantum gravity is to provide a description of the universe near the Big Bang singularity, in the regime where classical equations of general relativity break down. Within the LQG framework loop quantum cosmology (LQC) has seen the most development. In this approach one starts with a symmetry-reduced model on the classical level and then implements loop quantisation techniques to obtain a theory of (symmetry-reduced) quantum geometry. Another approach, that we take in this work, is to start with the full theory and apply it to a cosmological model. Given the spinfoam dynamics, quantum transition amplitudes can be calculated, giving rise to spinfoam cosmology. The definition and interpretation of transition amplitudes in a background-independent theory of quantum gravity is subtle: see, for example, a recent work on black hole dynamics [8]. Bianchi, Rovelli and Vidotto [9] studied transition amplitudes defined by the EPRL vertex and demonstrated that there is an approximation leading to the classical Friedmann-Robertson-Walker (FRW) cosmology.

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\* [ilya.vilensky@fau.edu](mailto:ilya.vilensky@fau.edu)

In the current paper we investigate quantum amplitudes using the Euclidean proper vertex. Thus, this work provides another test of the proper vertex which has been previously used (in its Lorentzian guise) to calculate the graviton propagator [10]. The calculation begins with fixing a graph. We choose the boundary states to be based on the graph with five nodes and ten links which can be viewed as a boundary of the 4-simplex. This boundary graph truncates the Hilbert space of the theory to a finite number of degrees of freedom. The boundary is seen to be a 3-dimensional slice of a homogeneous and isotropic universe. Then we pick as the boundary states the coherent states peaked on the intrinsic and extrinsic geometry of a spatial slice of FRW spacetime. These coherent states also encode the quantum fluctuations around the FRW geometry and their dynamics includes some inhomogeneous and anisotropic degrees of freedom.

We work at first order in the vertex expansion. The resulting quantum amplitude can be interpreted as the transition amplitude from a zero three-geometry to a compact three-geometry. Such amplitude has been proposed by Hartle and Hawking [11] as a quantum ground state of the universe, termed the Hartle-Hawking wavefunction. We proceed by evaluating the proper vertex amplitude in the coherent state representation. We estimate this amplitude in the large-volume limit. This allows us to use stationary phase methods to obtain an approximation for the Hartle-Hawking wavefunction.

To make a connection with the classical model we show that the amplitude  $W^{(+)}$  satisfies the operator constraint  $\hat{H}W^{(+)} = 0$ . We demonstrate that its classical analogue is the classical Hamiltonian constraint that arises in LQC. The dynamics of the model is found to select a particular family of coherent states. We shed light on this restriction by drawing an analogy with a similar dynamical selection in standard quantum mechanics.

The paper is organized as follows. In [Section II](#) the definitions of both EPRL and proper vertex amplitudes are reviewed. In [Section III](#) the approximations are presented and the amplitude is evaluated. In [Section IV](#) we analyse the classical limit and dynamical restrictions on the set of coherent states. We close with a summary of the results and a discussion of future work.

## II. PRELIMINARIES

### A. EPRL vertex

We recall the Spin(4) EPRL vertex amplitude defined on a given oriented 4-simplex. The tetrahedra have labels running from 0 to 4 which we denote  $a, b$ . The boundary Hilbert space is spanned by SU(2) generalised spin network states  $\Psi$  labelled by spins  $j_{ab}$  and vectors  $\psi_{ab}, \psi_{ba}$  in the corresponding irreducible representation of SU(2), defined explicitly by  $\Psi(\{U_{ab}\}) = \prod_{a < b} \langle \psi_{ab} | U_{ab} | \psi_{ba} \rangle$  with  $a, b$  taking values in the range from 0 to 4.

Let  $V_j$  denote the representation space for the spin  $j$  representation of SU(2) which will be denoted by  $\rho_j(g)$  for  $g \in \text{SU}(2)$  (the  $j$  subscript will be omitted when it is clear from the context).

Let  $\hat{L}^i$  denote the generators in each of these representations. Let  $\epsilon : V_j \times V_j \rightarrow \mathbb{C}$  be the invariant bilinear inner product and  $\langle \cdot, \cdot \rangle$  the Hermitian inner product on  $V_j$ . An antilinear structure map  $J : V_j \rightarrow V_j$  is then given by  $\epsilon(\psi, \phi) = \langle J\psi, \phi \rangle$ .  $J$  commutes with the group representation matrices and anticommutes with the generators.

Now let  $V_{j^+, j^-} = V_{j^+} \otimes V_{j^-}$  denote the representation space for the spin  $(j^+, j^-)$  representation of  $\text{Spin}(4) = \text{SU}(2) \times \text{SU}(2)$  and  $\rho_{j^+, j^-}(X^+, X^-) := \rho_{j^+}(X^+) \otimes \rho_{j^-}(X^-)$  denote the representation of  $(X^+, X^-) \in \text{Spin}(4)$  (again with the subscripts dropped when clear from the context). Define the bilinear form  $\epsilon : V_{j^+, j^-} \times V_{j^+, j^-} \rightarrow \mathbb{C}$  by  $\epsilon(\psi^+ \otimes \psi^-, \phi^+ \otimes \phi^-) := \epsilon(\psi^+, \phi^+) \epsilon(\psi^-, \phi^-)$  and the antilinear map  $J : V_{j^+, j^-} \rightarrow V_{j^+, j^-}$  by  $J(\psi^+ \otimes \psi^-) = (J\psi^+) \otimes (J\psi^-)$ . Then  $\epsilon(\Psi, \Phi) = \langle J\Psi, \Phi \rangle$ . Finally, let  $Y_j^{j^+, j^-} : V_j \rightarrow V_{j^+, j^-}$  denote the Clebsch-Gordan intertwining map.

A group element  $G_a = (X_a^+, X_a^-)$  is assigned to each tetrahedron  $a$  in the boundary of the 4-simplex. Define  $G_{ab} := (G_a)^{-1} G_b$ . This group element can be interpreted for each pair of tetrahedra  $a, b$  as the parallel transport map  $G_{ab} = (X_{ab}^+, X_{ab}^-)$  from the frame of tetrahedron  $b$  to the frame of tetrahedron  $a$ .

The imposition of the linear simplicity constraint fixes

$$j_{ab}^\pm = \frac{|1 \pm \gamma|}{2} j_{ab}$$

where  $\gamma$  is the Barbero-Immirzi parameter. Then the EPRL vertex amplitude for a given LQG boundary state  $\Psi_{\{j_{ab}, \psi_{ab}\}}$  is

$$A_v(\Psi_{\{j_{ab}, \psi_{ab}\}}) = \int_{\text{Spin}(4)^5} \prod_a dG_a \prod_{a < b} \epsilon(Y_{j_{ab}}^{j_{ab}^+, j_{ab}^-} \psi_{ab}, \rho(G_{ab}) Y_{j_{ab}}^{j_{ab}^+, j_{ab}^-} \psi_{ba}). \quad (2.1)$$

In this paper we will use the coherent state formulation of the vertex amplitude where instead of the vectors  $\psi_{ab}, \psi_{ba}$  the boundary spin-network states are labelled by the Perelomov coherent states [12]  $C_{\xi_{ab}}^{j_{ab}}, C_{\xi_{ba}}^{j_{ab}}$  associated with unit spinors  $\xi_{ab}, \xi_{ba}$ . We also define a unit 3-vector  $n_\xi$ , corresponding to a 2-spinor  $\xi$ , by

$$n_\xi := \frac{\langle \xi | \sigma | \xi \rangle}{\langle \xi | \xi \rangle}.$$

For any normalised spinor  $\xi$  take

$$g(\xi) = \begin{pmatrix} \xi_0 & -\bar{\xi}_1 \\ \xi_1 & \bar{\xi}_0 \end{pmatrix} \in \text{SU}(2).$$

Then the coherent state  $C_\xi^j$  is given by

$$C_\xi^j := g(\xi) |j, j\rangle,$$

that is, the highest weight eigenstate of  $n_\xi \cdot \hat{L}$  in the spin  $j$  representation. Therefore, the EPRL vertex amplitude on the coherent states is

$$A_v(\{j_{ab}, C_{\xi_{ab}}^{j_{ab}}\}) = \int_{\text{Spin}(4)^5} \prod_a dG_a \prod_{a < b} \epsilon(Y_{j_{ab}}^{j_{ab}^+, j_{ab}^-} C_{\xi_{ab}}^{j_{ab}}, \rho(G_{ab}) Y_{j_{ab}}^{j_{ab}^+, j_{ab}^-} C_{\xi_{ba}}^{j_{ab}}). \quad (2.2)$$

## B. Proper vertex

When the boundary data defines a non-degenerate 4-simplex geometry, Barrett *et al.* show that the EPRL vertex amplitude contains four terms in the semi-classical limit [13]. In a series of papers [4–7] Engle introduced the proper vertex amplitude and showed that its semi-classical limit comprises only one term with the Regge action appearing with the positive sign. This amplitude is defined by

$$A_v^{(+)} = \int_{\text{Spin}(4)^5} \prod_a dG_a \prod_{a<b} \epsilon(Y_{jab}^{j_{ab}^+, j_{ab}^-} \psi_{ab}, \rho(G_{ab}) Y_{jab}^{j_{ab}^+, j_{ab}^-} \Pi_{ba}(\{G_{a'b'}\}) \psi_{ba}) \quad (2.3)$$

where  $\Pi_{ba}(\{G_{a'b'}\})$  is a projection operator acting in the spin  $j_{ab}$  representation of  $\text{SU}(2)$ , given by

$$\Pi_{ba}(\{G_{a'b'}\}) := \Pi_{(0,\infty)}(\beta_{ab}(\{G_{a'b'}\}) \text{tr}(\sigma_i X_{ab}^- X_{ba}^+) \hat{L}^i). \quad (2.4)$$

Here  $\Pi_{(0,\infty)}(\hat{O})$  denotes the spectral projector onto the positive part of the spectrum of the operator  $\hat{O}$ ,

$$\beta_{ab}(\{G_{a'b'}\}) = -\text{sgn}[\epsilon_{ijk} n_{ac}^i n_{ad}^j n_{ae}^k \epsilon_{lmn} n_{bc}^l n_{bd}^m n_{be}^n],$$

with  $\{c, d, e\} = \{0, \dots, 4\} \setminus \{a, b\}$ , and

$$n_{ab}^i = \text{tr}(\sigma^i X_{ab}^- X_{ba}^+).$$

The amplitude can be written using coherent states on the boundary as

$$A_v^{(+)}(\{j_{ab}, C_{\xi_{ab}}^{j_{ab}}\}) = \int_{\text{Spin}(4)^5} \prod_a dG_a \prod_{a<b} \epsilon(Y_{jab}^{j_{ab}^+, j_{ab}^-} C_{\xi_{ab}}^{j_{ab}}, \rho(G_{ab}) Y_{jab}^{j_{ab}^+, j_{ab}^-} \Pi_{ba}(\{G_{a'b'}\}) C_{\xi_{ba}}^{j_{ab}}). \quad (2.5)$$

## III. COSMOLOGICAL SET-UP

### A. Choice of a graph and the boundary states

Spinfoam vertex amplitudes give path-integral transition amplitudes for fixed boundary states. To perform this calculation we choose a graph thereby truncating the boundary Hilbert space. Specifically, we choose a graph  $\Gamma_5$  formed by five nodes connected with ten links (see Fig. 1). This graph can be seen as the boundary of a 4-simplex. It can be endowed with a geometrical interpretation as follows. Consider a compact connected 3-manifold  $M$  with the topology of a 3-sphere. Then the boundary of a 4-simplex can be viewed as a triangulation of  $M$ . The 3-manifold  $M$  represents a spatial slice of a homogeneous and isotropic universe.

The next step involves picking the boundary states. These LQG states should be peaked on both extrinsic and intrinsic geometry of the 3-manifold and, therefore, are superpositions of spin networks. Such states are known in the literature [14–16] and given explicitly by

$$\Psi_{H_l}(U_l) = \int_{\text{SU}(2)^N} dg_n \prod_l K_t(g_{s(l)}^{-1} U_l g_{t(l)}) H_l^{-1} \quad (3.1)$$

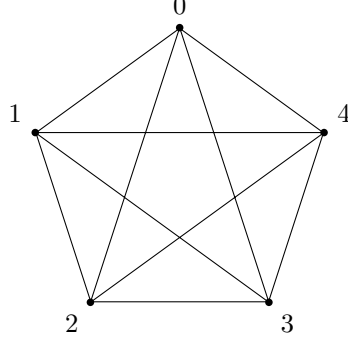


Fig. 1: The boundary graph  $\Gamma_5$ .

where  $K_t$  is the heat kernel function of the form

$$K_t(g) = \sum_j d_j e^{-thj(j+1)} \text{tr}(D^j(g)), \quad (3.2)$$

with  $d_j = 2j + 1$  and  $D^j(g)$  the Wigner matrix in the spin- $j$  representation of  $\text{SU}(2)$ . The labels  $H_l$  appearing above are elements of  $\text{SL}(2, \mathbb{C})$  and can be written as [17]

$$H_l = n_{s(l)} e^{-i(\xi_l + i\eta_l)(\sigma_3/2)} n_{t(l)}^{-1}, \quad (3.3)$$

with  $n_{s(l)}$ ,  $n_{t(l)}$  elements of  $\text{SU}(2)$ . In these definitions  $s(l)$ ,  $t(l)$  denote, respectively, the source node and the target node of the link  $l$  of the boundary graph.

As shown in [9], homogeneity and isotropy lead to  $n_{s(l)} = n_{t(l)} = n_l$  and  $\xi_l$ ,  $\eta_l$  being independent of  $l$ . Bianchi, Rovelli and Vidotto derive the relationship between the LQG conjugate variables  $A$ ,  $E$  and the boundary state labels  $\xi$ ,  $\eta$ . Specifically, after identification of the 3-manifold  $M$  with the group manifold of  $\text{SU}(2)$ , the Killing form  $\hat{q}_{ab}$  can be viewed as the fiducial metric and left-invariant vector fields on  $\text{SU}(2)$  as the fiducial triads  $\hat{e}$  (with  $\hat{\omega}$  the corresponding co-triads). Then,<sup>1</sup>

$$A = c \hat{V}^{-1/3} \hat{\omega} \quad E = p \hat{V}^{-2/3} \sqrt{\hat{q}} \hat{e}, \quad (3.4)$$

with  $\hat{V}$  the fiducial volume and  $\hat{q}$  the determinant of the fiducial metric, and [9]

$$\xi_l = \xi = \alpha c \quad \eta_l = \eta = \beta p, \quad (3.5)$$

with  $\alpha$ ,  $\beta$  certain constants. Thus, the homogeneous and isotropic boundary states can be labelled equivalently by  $\xi$ ,  $\eta$  or  $c$ ,  $p$ .

Introducing the holomorphic variable  $z$

$$z = \xi + i\eta, \quad (3.6)$$

we can write

$$H_l = n_l e^{-iz\sigma_3/2} n_l^{-1}. \quad (3.7)$$

<sup>1</sup> In [9] the authors use slightly different definitions of  $(c, p)$ , here we use the definitions standard in LQC. The extra factors can be absorbed into constants  $\alpha$ ,  $\beta$ .

### B. Hartle-Hawking wavefunction

In [11] Hartle and Hawking proposed that the wavefunction for a three-geometry is given by the path integral over all compact four-geometries with this three-geometry as a boundary. Spinfoam dynamics of LQG boundary states allows us to implement this proposal. Specifically, we consider an amplitude given by the spinfoam formed from a single vertex bounded by five edges (see Fig. 2).

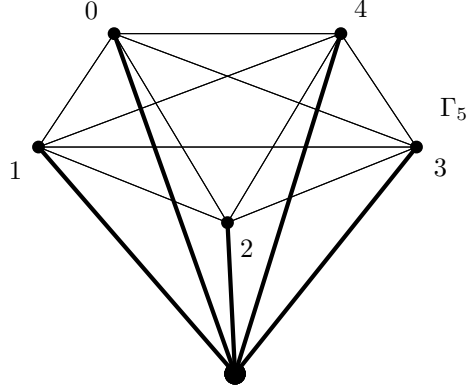


Fig. 2: Spinfoam with a single vertex.

Such an amplitude can be written as a holomorphic function of  $z$

$$W^{(+)}(z) = \langle W^{(+)} | \Psi_{H_{ab}(z)} \rangle \quad (3.8)$$

where  $W^{(+)}$  indicates the use of the proper vertex amplitude  $A_v^{(+)}$ . The links of the boundary graph are now labelled by pairs of indices  $(ab)$ , with  $a, b$  denoting the nodes of the graph corresponding to the tetrahedra as described in Section II. We note here that this particular choice of the spinfoam is motivated partly by the fact that the proper vertex amplitude has so far only been defined for 2-complexes dual to triangulations made up of 4-simplices.

The amplitude  $W^{(+)}(z)$  can be viewed as a transition amplitude from a zero three-geometry (a single point) to the three-geometry specified by  $z$  (with a finite scale factor and extrinsic curvature). It can be rewritten as

$$W^{(+)}(z) = \int_{\text{SU}(2)^{10}} dU_{ab} W^{(+)}(U_{ab}) \Psi_{H_{ab}(z)}(U_{ab}). \quad (3.9)$$

### C. Large volume limit

We will calculate the amplitude (3.8) in the large volume limit. This limit is obtained by taking  $p$  large or equivalently considering  $\eta \gg 1$ . Using (3.7), we write

$$D^j(H_l) = D^j(n_l) D^j(e^{-iz\sigma_3/2}) D^j(n_l^{-1}). \quad (3.10)$$

In the large  $\eta$  limit, we then have [16]

$$D^j \left( e^{-iz\sigma_3/2} \right) \approx e^{-izj} |j, j\rangle \langle j, j|. \quad (3.11)$$

Therefore, rewriting (3.1), we get

$$\Psi_{H_l}(U_l) \approx \sum_{j_l} \left( \prod_l d_{j_l} e^{-t\hbar j_l(j_l+1) - izj_l} \right) \int_{\text{SU}(2)^N} dh_n \prod_l \langle C_{\xi_l}^{j_l} | h_{s(l)}^{-1} U_l h_{t(l)} | C_{\xi_l}^{j_l} \rangle. \quad (3.12)$$

Here the unit spinors  $\xi_l$  are chosen to satisfy  $n_{\xi_l} = n_l$ .

Plugging this expression into (3.9), so that the links of the graph are now labelled by  $(ab)$  instead of  $l$ , performing integrals over  $U_{ab}$  and using the invariance of  $\text{Spin}(4)$  measure, we obtain

$$W^{(+)}(z) = \sum_{j_{ab}} \left( \prod_{a<b} d_{j_{ab}} e^{-t\hbar j_{ab}(j_{ab}+1) - izj_{ab}} \right) A_v^{(+)} \left( \{j_{ab}, C_{\xi_{ab}}^{j_{ab}}\} \right). \quad (3.13)$$

In the limit  $\eta \gg 1$  the Gaussian form of the prefactor picks out large values of  $j_{ab}$ . Therefore, we can evaluate the amplitude factor in the large spin limit. Here the large spin limit is taken by setting all ten spins equal  $j_{ab} = j$  and scaling  $j \rightarrow \lambda j = j_0$ . We will use the extended stationary phase theorem to obtain the asymptotic limit for large  $\lambda$ .

To apply stationary phase methods, we first rewrite the amplitude in an exponentiated form. Inserting the completeness relation for coherent states  $C_\eta^j$  into (2.5), we obtain:

$$A_v^{(+)} \left( \{j_{ab}, C_{\xi_{ab}}^{j_{ab}}\} \right) = \int_{\text{Spin}(4)^5} \prod_a dG_a \int_{\mathbb{CP}^{10}} \prod_{a<b} d\mu_{\eta_{ba}} \epsilon(Y_{j_{ab}}^{j_{ab}^+, j_{ab}^-} C_{\xi_{ab}}^{j_{ab}}, \rho(G_{ab}) Y_{j_{ab}}^{j_{ab}^+, j_{ab}^-} C_{\eta_{ba}}^{j_{ab}}) \langle C_{\eta_{ba}}^{j_{ab}} | \Pi_{ba}(\{G_{a'b'}\}) C_{\xi_{ba}}^{j_{ab}} \rangle \quad (3.14)$$

where  $\eta_{ba}$  are unit spinors (that is, for each of the ten pairs  $(ba)$  we have  $\langle \eta_{ba} | \eta_{ba} \rangle = 1$ ) and  $d\mu_{\eta_{ba}} = \frac{d_{j_{ab}}}{\pi} \Omega_{\eta_{ba}}$  with  $\Omega_{\eta_{ba}} = \frac{i}{2} (\epsilon_{AB} \eta_{ba}^A d\eta_{ba}^B) \wedge (\epsilon_{AB} \bar{\eta}_{ba}^A d\bar{\eta}_{ba}^B)$ . Then, introducing

$$S^{\text{EPRL}} = \sum_{a<b} \log \epsilon(Y_{j_{ab}}^{j_{ab}^+, j_{ab}^-} C_{\xi_{ab}}^{j_{ab}}, \rho(G_{ab}) Y_{j_{ab}}^{j_{ab}^+, j_{ab}^-} C_{\eta_{ba}}^{j_{ab}}) \quad (3.15)$$

$$S^\Pi = \sum_{a<b} S_{ab}^\Pi = \sum_{a<b} \log \langle C_{\eta_{ba}}^{j_{ab}} | \Pi_{ba}(\{G_{a'b'}\}) C_{\xi_{ba}}^{j_{ab}} \rangle \quad (3.16)$$

$$S^{(+)} = S^{\text{EPRL}} + S^\Pi, \quad (3.17)$$

we write the amplitude as

$$A_v^{(+)} \left( \{j_{ab}, C_{\xi_{ab}}^{j_{ab}}\} \right) = \int_{\text{Spin}(4)^5} \prod_a dG_a \int_{\mathbb{CP}^{10}} \prod_{a<b} d\mu_{\eta_{ba}} e^{S^{(+)}}. \quad (3.18)$$

At this point the reader might expect us to proceed to calculate stationary points of the action. However, we still have to show that stationary phase methods are applicable in this case. The stumbling point is the fact that, while  $S^{\text{EPRL}}$  scales linearly with spins  $j_{ab}$  (see [13]),  $S^\Pi$  does not. In what follows, we show that  $S^\Pi$  is *asymptotically linear* in spins. We employ a strategy similar to the one applied by the author and his collaborators in [18] in the case of the Lorentzian proper vertex amplitude.

Let  $n_{\nu_{ba}} = \beta_{ab}(\{G_{a'b'}\}) \frac{\text{tr}(\sigma^i X_{ab}^- X_{ba}^+)}{|\text{tr}(\sigma^i X_{ab}^- X_{ba}^+)|}$  and  $\nu_{ba}$  be the corresponding unit spinor. Define

$$|\xi; k, m\rangle = g(\xi)|k, m\rangle. \quad (3.19)$$

The projector  $\Pi_{ba}(\{G_{a'b'}\})$  can be written explicitly

$$\Pi_{ba}(\{G_{a'b'}\}) = \Pi_{(0,\infty)}(n_{\nu_{ba}} \cdot \hat{L}) = \sum_{m>0}^{j_{ab}} |\nu_{ba}; j_{ab}, m\rangle \langle \nu_{ba}; j_{ab}, m|. \quad (3.20)$$

Then,

$$\begin{aligned} e^{S_{ab}^\Pi} &= \langle C_{\eta_{ba}}^{j_{ab}} | \Pi_{ba}(\{G_{a'b'}\}) C_{\xi_{ba}}^{j_{ab}} \rangle \\ &= \sum_{m>0}^{j_{ab}} \langle j_{ab}, j_{ab} | g(\eta_{ba})^{-1} g(\nu_{ba}) | j_{ab}, m \rangle \langle j_{ab}, m | g(\nu_{ba})^{-1} g(\xi_{ba}) | j_{ab}, j_{ab} \rangle \end{aligned} \quad (3.21)$$

From this we can use exactly the same argument as in the Lorentzian proper vertex asymptotics paper and obtain (for the details of the argument see [18])<sup>2</sup>

$$\exp(S_{ab}^\Pi) \sim \begin{cases} (x_{ab} + y_{ab})^{2\lambda_j} & \text{if } |x_{ab}| > |y_{ab}| \text{ and } |x_{ab} + y_{ab}|^2 \geq |4x_{ab}y_{ab}| \\ \frac{(4x_{ab}y_{ab})^{\lambda_j}}{\sqrt{\pi\lambda_j}} \frac{x_{ab}}{y_{ab} - x_{ab}} & \text{if } |x_{ab}| < |y_{ab}| \text{ or } |x_{ab} + y_{ab}|^2 < |4x_{ab}y_{ab}| \end{cases} \quad (3.22)$$

where  $x_{ab} := \langle \eta_{ba}, \nu_{ba} \rangle \langle \nu_{ba}, \xi_{ba} \rangle$  and  $y_{ab} := \langle \eta_{ba}, J\nu_{ba} \rangle \langle J\nu_{ba}, \xi_{ba} \rangle$ .

Using lemma 4 and theorem 4 in [18] and the analysis of the critical points of the action in [7] we deduce the asymptotics:

$$A_v^{(+)}(\{j, C_{\xi_{ab}}^j\}) \Big|_{j \rightarrow j_0} \sim j^{-12} N^{(+)} \exp\left(i \sum_{a<b} \gamma j \Theta_{ab}\right) \Big|_{j \rightarrow j_0}. \quad (3.23)$$

Here  $N^{(+)}$  is independent of  $j$  and  $\Theta_{ab}$  are dihedral angles determined by  $N_a \cdot N_b = \cos \Theta_{ab}$  with  $N_a, N_b$  the outward normals to the  $a$  and  $b$  tetrahedra, respectively. In the case of the regular 4-simplex, considered in this paper,  $\Theta_{ab} \equiv \Theta := \arccos(-\frac{1}{4})$ .

Using this asymptotics in (3.13) and defining  $\tilde{z} := z - \gamma\Theta$ , we obtain

$$W^{(+)}(z) = \sum_{j_{ab}} \left( \prod_{a<b} d_{j_{ab}} e^{-th_{j_{ab}}(j_{ab}+1) - i\tilde{z}j_{ab}} \right) N_{\{j_{ab}\}} \quad (3.24)$$

with

$$N_{\{j_{ab}\}} = j_{ab}^{-12} N^{(+)} \Big|_{j_{ab} \rightarrow j_0}. \quad (3.25)$$

We can write the prefactor in the explicitly Gaussian form as

$$W^{(+)}(z) \approx \sum_{j_{ab}} \left( \prod_{a<b} d_{j_{ab}} e^{-\frac{\hbar(j_{ab}-j_0)^2}{2\sigma^2}} e^{-\frac{\tilde{z}^2 \sigma^2}{2\hbar}} \right) N_{\{j_{ab}\}} \quad (3.26)$$

<sup>2</sup> There is a subtlety here in that these expressions apply to the case of integer  $j$ . However, similar expressions would pertain in the half-integer case and, in fact, the resulting asymptotics of the vertex amplitude (3.23) is exactly the same in both cases.



with  $j_0 = -\frac{i\tilde{z}}{2t\hbar}$  and  $\sigma = \frac{1}{\sqrt{2t}}$ . Given that  $\text{Re}(j_0) \sim \eta$ , in the large volume limit we can therefore approximate the sum by an integral. Performing the integration, we get

$$W^{(+)}(z) = \left( \sqrt{\frac{2\pi\sigma^2}{\hbar}} 2j_0 e^{-\frac{\tilde{z}^2\sigma^2}{2\hbar}} \right)^{10} j_0^{-12} N^{(+)}, \quad (3.27)$$

and, substituting the definition of  $j_0$ , we obtain

$$W^{(+)}(z) \approx N \tilde{z}^{-2} e^{\frac{-5\tilde{z}^2\sigma^2}{\hbar}} \quad (3.28)$$

where  $N = -(8\pi)^5 (\sigma^3 \hbar)^2 N^{(+)}$ . This is the Hartle-Hawking wavefunction of a closed, homogeneous and isotropic universe.

#### IV. CLASSICAL LIMIT

In this section we will confirm that the Hartle-Hawking state (3.28) satisfies the Hamiltonian constraint in the classical limit. The Hamiltonian constraint in FRW models is given by [19]:

$$C_H = -\frac{3}{8\pi G\gamma^2} c^2 |p|^{\frac{1}{2}} \text{sgn}(p) = 0. \quad (4.1)$$

Rescaling by  $|p|^{\frac{3}{2}} \text{sgn}(p)$  we have

$$C_H = -\frac{3}{8\pi G\gamma^2} c^2 p^2 = 0. \quad (4.2)$$

Using (3.5) and (3.6),

$$C_H = \frac{3}{128\pi G\gamma^2 (\alpha\beta)^2} (z^2 - \bar{z}^2)^2. \quad (4.3)$$

Let us fix  $\sigma$  so that  $z$  is a coordinate in phase space with symplectic structure

$$\omega = 10i\sigma^2 dz \wedge d\bar{z}. \quad (4.4)$$

We will discuss below the significance of this choice. Then the Poisson bracket reads  $\{z, \bar{z}\} = \frac{i}{10\sigma^2}$ . Choosing a holomorphic polarisation for the quantisation, we get the states as holomorphic functions of  $z$  and, bearing in mind that  $[\hat{z}, \hat{\bar{z}}] = i\hbar\{z, \bar{z}\}$ , we define the quantisation of the phase space variables to be

$$\hat{z} = z \quad \hat{\bar{z}} = \frac{\hbar}{10\sigma^2} \frac{d}{dz} - \gamma\Theta \quad (4.5)$$

These operators satisfy the commutation relations. Furthermore, the adjointness condition  $\hat{\bar{z}}^\dagger = \hat{z}$  is fulfilled when the Hermitian inner product on Hilbert space is taken to be

$$\langle \psi, \phi \rangle = \int e^{-\frac{10\sigma^2}{\hbar}(|z|^2 + 2\gamma\Theta \text{Re } z)} \bar{\psi} \phi d^2 z. \quad (4.6)$$

If we now specify an operator  $\hat{H}$  as

$$\hat{H} = \frac{3}{128\pi G\gamma^2(\alpha\beta)^2} \left( z^2 - \left( \frac{\hbar}{10\sigma^2} \frac{d}{dz} - \gamma\Theta \right)^2 + \frac{3}{10}\hbar\sigma^{-2} + \frac{2}{5}\hbar\sigma^{-2}\gamma\Theta\bar{z}^{-1} + \frac{3}{50}\hbar^2\sigma^{-4}\bar{z}^{-2} \right)^2, \quad (4.7)$$

we can note that

$$\hat{H}W^{(+)}(z) = 0. \quad (4.8)$$

Considering the limit  $\hbar \rightarrow 0$ , we see that the classical analogue of  $\hat{H}$  is  $C_H$ . Therefore, we deduce that  $\hat{H}$  is a possible quantisation of  $C_H$ . We can conclude that the Hartle-Hawking wavefunction  $W^{(+)}(z)$  describes a state (with the interpretation of the holomorphic variable  $z$  as the coordinate in the phase space defined above) in a quantisation of the FRW model that satisfies the Hamiltonian constraint.

Let us comment on the fact that we had to fix  $\sigma$  for this calculation. The choice of  $\sigma$  (or, equivalently, heat kernel time  $t$ ) amounts to selecting a specific family of coherent states as the boundary states for the amplitude. We can interpret this choice as a restriction imposed by the dynamics of the problem on the allowable set of coherent states. Such dynamic restrictions on coherent states arise in standard quantum mechanics [20]. In what follows we illustrate this with a simple example.

Consider a quantum harmonic oscillator specified by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2}. \quad (4.9)$$

Define the creation and annihilation operators  $\hat{a}_\kappa^\dagger, \hat{a}_\kappa$ :

$$\hat{a}_\kappa^\dagger = \sqrt{\frac{\kappa}{2\hbar}} \hat{x} - i\sqrt{\frac{1}{2\hbar\kappa}} \hat{p} \quad \hat{a}_\kappa = \sqrt{\frac{\kappa}{2\hbar}} \hat{x} + i\sqrt{\frac{1}{2\hbar\kappa}} \hat{p} \quad (4.10)$$

The vacuum state  $|0_\kappa\rangle$  is annihilated by the annihilation operator:

$$\hat{a}_\kappa|0_\kappa\rangle = 0. \quad (4.11)$$

Let  $\hat{D}_\kappa(\alpha)$  be the unitary displacement operator and define coherent states

$$|\psi_\kappa\rangle := \hat{D}_\kappa(\alpha)|0_\kappa\rangle = e^{\alpha\hat{a}_\kappa^\dagger - \bar{\alpha}\hat{a}_\kappa}|0_\kappa\rangle. \quad (4.12)$$

These coherent states are eigenstates of the annihilation operator,

$$\hat{a}_\kappa|\psi_\kappa\rangle = \alpha|\psi_\kappa\rangle, \quad (4.13)$$

and saturate the lower bound for the product of uncertainties  $(\Delta x)(\Delta p)$  given by the Heisenberg uncertainty principle. These families of coherent states are characterised by the parameter  $\kappa$ .

It is the dynamics of the harmonic oscillator that fixes the parameter  $\kappa$ . Using (4.11) and (4.12) and making  $\alpha$  time-dependent, we can define a time-dependent normalised coherent state  $\psi_\kappa(x, t)$ :

$$\psi_\kappa(x, t) := \left( \frac{\kappa}{\pi\hbar} \right)^{\frac{1}{4}} e^{-\frac{\kappa}{2\hbar}x^2 + \sqrt{\frac{2\kappa}{\hbar}}\alpha(t)x - \alpha(t)\text{Re}\alpha(t) + i\phi(t)} \quad (4.14)$$

where  $\phi(t)$  is a phase factor. We now impose that  $\psi_\kappa(t)$  satisfy the Schrödinger equation:

$$\hat{H}|\psi_\kappa(t)\rangle = i\hbar \frac{d}{dt}|\psi_\kappa(t)\rangle. \quad (4.15)$$

This yields for the harmonic oscillator characterised by the Hamiltonian (4.9):

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{m\omega^2 x^2}{2}\right) \psi_\kappa(x, t) = i\hbar \frac{d}{dt} \psi_\kappa(x, t). \quad (4.16)$$

Solving this condition gives  $\kappa = m\omega$ . Hence, the dynamics of the problem restricts the family of the coherent states to the canonical coherent states associated with the quantum harmonic oscillator.

Similarly to (4.15), in the case of quantum cosmology the condition being imposed reads:

$$\hat{H}|\Psi_\sigma\rangle = 0. \quad (4.17)$$

This condition, as we have seen, likewise selects a specific family of heat-kernel coherent states by fixing  $\sigma$  and, therefore, heat-kernel time  $t$ .

## V. CONCLUSION

In this work we investigated spinfoam cosmology using the recently introduced proper vertex amplitude. We evaluated the Hartle-Hawking wavefunction as a spinfoam transition amplitude from a zero three-geometry to a finite three-geometry. To perform the calculation we introduced a fixed graph thereby truncating the boundary Hilbert space of the theory. On this graph we considered coherent boundary states peaked on the classical spatial geometry and extrinsic curvature of the FRW model. The spinfoam expansion was approximated by a single proper vertex. We analysed the asymptotics of the proper vertex for large spins and obtained the associated transition amplitude in the large volume limit.

The amplitude turned out to satisfy an operator constraint. This operator constraint can be viewed as a quantisation of the classical Hamiltonian constraint arising in LQC. Note, however, that the dynamics is rather trivial: the spacetime is flat, which is the unique non-degenerate solution of Einstein's equations in the absence of matter and cosmological constant. We found that the dynamics imposes a restriction on the relevant family of coherent states. This is not surprising, because such restrictions arise in standard quantum mechanics. We demonstrated a similar coherent state selection on the example of a quantum harmonic oscillator.

There are multiple avenues for further investigations: one could include matter or cosmological constant (see [21, 22] for previous work in the EPRL model). One could consider larger graphs and higher orders in the vertex expansion to check the validity of approximations. Another task would be to apply the Lorentzian proper vertex to spinfoam cosmology. Since by construction the boundary data is that of a Euclidean 4-simplex, only critical points in the degenerate sector are selected [23]. Therefore, the corresponding contributions are expected to be suppressed in the asymptotics of the proper vertex [18], and the calculation will have to include the next order in the vertex expansion.

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- [1] J. Engle, R. Pereira, and C. Rovelli, *Phys. Rev. Lett.* **99**, 161301 (2007), [arXiv:0705.2388 \[gr-qc\]](#).
  - [2] J. Engle, R. Pereira, and C. Rovelli, *Nucl. Phys. B* **798**, 251 (2008), [arXiv:0708.1236 \[gr-qc\]](#).
  - [3] J. Engle, E. Livine, R. Pereira, and C. Rovelli, *Nucl. Phys. B* **799**, 136 (2008), [arXiv:0711.0146 \[gr-qc\]](#).
  - [4] J. Engle, *Class. Quantum Grav.* **30**, 049501 (2013), [arXiv:1301.2214 \[gr-qc\]](#).
  - [5] J. Engle, *Class. Quantum Grav.* **28**, 225003 (2011), [arXiv:1107.0709 \[gr-qc\]](#).
  - [6] J. Engle, *Phys. Lett. B* **724**, 333 (2013), [arXiv:1201.2187 \[gr-qc\]](#).
  - [7] J. Engle, *Phys. Rev. D* **87**, 084048 (2013), [arXiv:1111.2865 \[gr-qc\]](#).
  - [8] M. Christodoulou, C. Rovelli, S. Speziale, and I. Vilenky, *Phys. Rev. D* **94**, 084035 (2016), [arXiv:1605.05268 \[gr-qc\]](#).
  - [9] E. Bianchi, C. Rovelli, and F. Vidotto, *Phys. Rev. D* **82**, 084035 (2010), [arXiv:1003.3483 \[gr-qc\]](#).
  - [10] A. Chaharsough Shirazi, J. Engle, and I. Vilenky, *Class. Quantum Grav.* **33**, 205010 (2016), [arXiv:1511.03644 \[gr-qc\]](#).
  - [11] J. Hartle and S. Hawking, *Phys. Rev. D* **28**, 2960 (1983).
  - [12] A. Perelomov, *Generalized Coherent States and Their Applications* (Springer, Berlin, 1986).
  - [13] J. W. Barrett, R. Dowdall, W. J. Fairbairn, H. Gomes, and F. Hellmann, *J. Math. Phys.* **50**, 112504 (2009), [arXiv:0902.1170 \[gr-qc\]](#).
  - [14] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao, and T. Thiemann, *J. Funct. Anal.* **135**, 519 (1996), [arXiv:gr-qc/9412014 \[gr-qc\]](#).
  - [15] T. Thiemann, *Class. Quantum Grav.* **23**, 2063 (2006), [arXiv:gr-qc/0206037](#).
  - [16] E. Bianchi, E. Magliaro, and C. Perini, *Phys. Rev. D* **82**, 024012 (2009), [arXiv:0912.4054 \[gr-qc\]](#).
  - [17] L. Freidel and S. Speziale, *Phys. Rev. D* **82**, 084040 (2010).
  - [18] J. Engle, I. Vilenky, and A. Zipfel, *Phys. Rev. D* **94**, 064025 (2016), [arXiv:1505.06683 \[gr-qc\]](#).
  - [19] A. Ashtekar, M. Bojowald, and J. Lewandowski, *Adv. Theor. Math. Phys.* **7**, 233 (2003).
  - [20] J. P. Gazeau and J. R. Klauder, *J. Phys. A: Math. Gen.* **32**, 123 (1999).
  - [21] E. Bianchi, T. Krajewski, C. Rovelli, and F. Vidotto, *Phys. Rev. D* **83**, 104015 (2011).
  - [22] F. Vidotto, *Class. Quantum Grav.* **28**, 245005 (2011).
  - [23] J. Barrett, R. Dowdall, W. Fairbairn, F. Hellmann, and R. Pereira, *Class. Quantum Grav.* **27**, 165009 (2010).